Traveling kinks in discrete media: Exact solution in a piecewise linear model

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We present an exact integral representation of a traveling kink solution in a reaction-diffusion equation with a piecewise linear reaction function, complementing existence proofs and numerical observations of such solutions in discrete excitable media. The kink speed is determined through a matching condition, and is worked out explicitly in two limiting situations: the pinning limit, and the opposite limit of infinitely fast kink. Results on the pinning limit agree with those in a recent paper by Fath [Physica D **116**, 176 (1998)]. The model includes a ''recovery parameter'' for a possible extension to a discrete FitzHugh-Nagumo-type system.

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Nonlinear reaction-diffusion equations are used widely to model *excitable media* and the rich variety of patterns, kinks, pulses, and waves that such media support. While an extensive literature exists relating to these objects in one-dimensional (1D), 2D, and 3D continuous media, spatially *discrete* excitable systems are less well-studied by comparison. It appears worthwhile to explore special features of patterns and waves that show up in discrete models by contrast to continuous ones. As an analogy pertaining to a different (nondissipative) context, one may point to the so-called discrete breather solution that has been shown to occur routinely in discrete Hamiltonian lattices but is known to be rare in continuous systems (for a review, see [1,2]). One possible novelty in discrete dissipative systems relates to the *pinning* of traveling kinks and pulses on a lattice (pinning is known to be a feature of localized excitations in nondissipative discrete media as well). This feature of discrete reaction-diffusion systems has been fruitfully invoked to understand diverse physical and biological phenomena [3–7]. In this paper we consider a 1D lattice with a scalar reaction variable u_n at lattice site n ($n=0,\pm1,\pm2,\ldots$), the time evolution of the u_n's being described by the reaction-diffusion system (discrete Nagumo equation)

$$\frac{du_n}{dt}=D(u_{n+1}-2u_n+u_{n-1})+f(u_n) \quad (D>0), \tag{1}$$

where D is a coefficient coupling neighboring lattice sites and $f(u_n)$ is a nonlinear bistable reaction function that we specify below.

Zinner [8] gave a rigorous proof for the existence of traveling front solutions of Eq. (1) for appropriate values of the coupling constant D. Keener [9] investigated the phenomenon of pinning, or propagation failure in the context of excitability of the heart muscle, while Erneux and Nicolis [10] studied the critical behavior near the pinning transition. More recently, Fath [11] investigated a discrete Nagumo-like model with a piecewise linear (PWL) reaction function $f(u)$, describing a number of exact limiting characteristics of the traveling front (''kink'') solution near the pinning transition in a perturbative analysis. In the following, we consider a similar PWL version of the discrete Nagumo equation and generalize Fath's work by way of constructing an *exact integral expression* for the traveling kink solution not only near the pinning transition but in the *entire* range of its existence. While the existence of traveling kink solutions in discrete reaction-diffusion equations has been rigorously demonstrated, there does not exist in the literature any exact *construction* of the traveling kink solution. We bridge this gap in the present paper in the framework of the PWL version. Starting from the exact integral expression, we investigate the two limiting situations corresponding to the speed going to zero (pinning transition) and infinity (vanishingly small threshold), respectively, and derive in the former situation the principal results obtained by Fath. The latter, on the other hand, gives a result relating to the speed of the traveling kink that our exact solution yields. The model additionally includes a ''recovery variable'' w for a possible extension to a more complete FitzHugh-Nagumo-type model in a discrete medium. Results on such extension will be reported elsewhere.

More precisely, we consider the Nagumo model (1) with the piecewise linear reaction function

$$f(u)=-u-w+\Theta(u-a), \tag{2}$$

where the ''threshold'' parameter a and the recovery parameter w satisfy

$$0<a<\tilde{a}, \tag{3a}$$

$$-a<w<1-a, \tag{3b}$$

and Θ stands for the Heaviside step function, while $\tilde{a}$ is a limiting value of the threshold parameter a (see below). In a more complete model, w should be a dynamical variable, and not simply a parameter, and the traveling front would be replaced by a traveling *pulse* solution. For the present, however, w is a parameter satisfying inequality (3b) and we seek a traveling kink solution of the form

$$u_n(t)=g(\zeta), \tag{4a}$$

$$\zeta=\chi t+n. \tag{4b}$$

We look upon χ as the ''speed'' of the kink and calculate it in terms of the parameters of the model while at the same time obtaining the function g describing the profile of the kink in terms of the kink variable ζ. In the context of any specific solution, and with f given by the PWL form (2), Eq. (1) is actually a linear equation for the kink variable ζ lying in specific intervals. The transition from any one of these intervals to an adjacent one corresponds to at least one of the u_n's crossing the value $u_n=a$. Hence, in such an interval, the desired solution $u_n(t)$ can be written as a linear combination of *basic* solutions.The kink solution we seek corresponds to

$$g(\zeta)>a \text{ for } \zeta>0, \tag{5a}$$

$$g(\zeta)=a \text{ for } \zeta=0, \tag{5b}$$

$$g(\zeta)<a \text{ for } \zeta<0. \tag{5c}$$

This means, in particular,

$$u_n(0)>a \text{ for } n>0, \tag{6a}$$

$$u_n(0)=a \text{ for } n=0, \tag{6b}$$

$$u_n(0)<a \text{ for } n<0. \tag{6c}$$

Moreover, for $0<t<1/\chi$, we have

$$f(u_n)=-u_n-w+1 \quad (n\geqslant 0), \tag{7a}$$

or

$$=-u_n-w \quad (n<0). \tag{7b}$$

One can now obtain the basic solutions in this time interval and obtain a superposition such that, on using appropriate matching conditions and on transforming suitably from t, n to the kink variable ζ, the required kink solution is arrived at. Using Eqs. (7a) and (7b) in Eq. (1) one obtains a linear inhomogeneous system of equations for the time interval $0<t<1/\chi$,

$$\frac{du_n}{dt}=D(u_{n+1}-2u_n+u_{n-1})-u_n-w+\Theta(n). \tag{8}$$

Thus the variables

$$v_n(t)=u_n-a_n, \tag{9a}$$

satisfy the homogeneous system

$$\frac{dv_n}{dt}=D(v_{n+1}+v_{n-1})-(2D+1)v_n, \tag{9b}$$

where the a_n's satisfy

$$D(a_{n+1}-2a_n+a_{n-1})=a_n+w-\Theta(n), \tag{9c}$$

and constitute the fixed point of the linear system (8). It helps to recognize that the a_n's need not satisfy Eqs. (6a), (6b), and (6c), which, indeed, they cannot, since these would then constitute a stationary kink solution and would exclude a propagating kink (see [11]). The sequence a_n satisfying Eq. (9c) constitute a trajectory of an inhomogeneous mapping and can be described in terms of a sequence of transfer matrices. As an alternative approach, one has a matrix equation

$$A=T^{-1}W, \tag{10}$$

where A is a bi-infinite column

$$A=\text{col}(\ldots a_{-2},a_{-1},a_0,a_1,a_2,\ldots), \tag{11}$$

W is a similar bi-infinite column, with

$$W_n=-1+w \quad (n\geqslant 0), \tag{12a}$$

or

$$=w \quad (n<0), \tag{12b}$$

and the matrix T has elements T_{ij} given by

$$T_{ii}=-(2D+1), \quad T_{i,i-1}=T_{i-1,i}=D \quad (i=0,\pm 1,\pm 2,\ldots), \tag{13}$$

all other T_{ij}'s being zero. It is a simple matter to invert this tridiagonal matrix subject to the requirement that elements of T^{-1} do not increase unboundedly, a requirement that corresponds to the boundary conditions

$$a_p\to(1-w) \text{ as } p\to\infty \tag{14a}$$

$$a_p\to -w \text{ as } p\to-\infty, \tag{14b}$$

consistent with the kink solution we are looking for. The elements of T^{-1} are given by

$$(T^{-1})_{i,i-k}=(T^{-1})_{i-k,i}=-\frac{1-\gamma}{1+\gamma}\gamma^k \quad (k=0,1,2,\ldots), \tag{15}$$

where γ is the solution of

$$\gamma^2-\left(2+\frac{1}{D}\right)\gamma+1=0, \tag{16}$$

satisfying $|\gamma|<1$ [recall from Eq. (1) that $D>0$], i.e.,

$$\gamma=1+\frac{1}{2D}-\sqrt{\left(\frac{1}{D}+\frac{1}{4D^2}\right)}. \tag{17}$$

With T^{-1} obtained above, Eq. (10) gives the column A

$$a_p=1-w-\frac{\gamma}{1+\gamma}\gamma^p \quad (p\geqslant 0), \tag{18a}$$

or,

$$=-w+\frac{1}{1+\gamma}\gamma^{-p} \quad (p<0). \tag{18b}$$

We now turn to Eq. (9b) and set up the basic solutions referred to above. A typical basic solution represents the linear evolution of a mode of the form $e^{in\theta}$. As can be seen from Eq. (9b), all such modes decay with time, with the decay rates depending on θ $(0 \leqslant \theta \leqslant \pi)$. In other words, a typical decaying mode looks as $e^{in\theta}e^{-\lambda(\theta)t}$ and the solution for $v_n(t)$ in any appropriate ζ interval involves a linear superposition of these. The full solution for $u_n(t)$ for all ζ would then be obtained on using Eq. (9a) and by matching together these "pieces." From Eq. (5) we note that, for instance, $u_{-1}(t=0)<a$ while $u_{-1}(t>1/\chi)>a$. In other words, as ζ changes by unity, some u_n or other crosses the value a. This means that the ζ intervals mentioned above are simply segments of unit length on the ζ axis, with $\zeta=0$ as an endpoint of one such segment, and in each of these intervals $g(\zeta)$ is a linear superposition indicated above. Using the reality of $g(\zeta)$ one is led to the following form:

$$g(\zeta)=a_p+\int_0^{\pi}[b(\theta)e^{ip\theta}+b^*(\theta)e^{-ip\theta}]e^{-\lambda(\theta)(\zeta-p)}d\theta, \tag{19}$$

where $p=[\zeta]$, the integer part of ζ, and $b(\theta)$ $(0 \leqslant \theta \leqslant \pi)$ is to be determined from matching conditions. In order to make Eq. (19) satisfy Eq. (1) we consider $u_0(t)$ for $p<\chi t(=\zeta)<(p+1)$. On substitution one finds that the decay rate $\lambda(\theta)$ satisfy

$$-\chi\lambda(\theta)=2D\cos\theta-(2D+1). \tag{20}$$

We now impose the boundary condition $g(0)=0$ (this just amounts to having the kink located at a chosen lattice point at $t=0$; in the present context this simplifies the calculation for obtaining the exact form of the kink) and the continuity conditions

$$g(n^-)=g(n^+) \quad (p=0,\pm1,\pm2,\ldots), \tag{21}$$

where (n^-) and (n^+) refer to values of ζ approaching n from the left and from the right, respectively. With a_p given by Eqs. (18a) and (18b), Eq. (19) gives $(p=0,\pm1,\pm2,\ldots)$

$$a_0+\int_0^{\pi}[b(\theta)+b^*(\theta)]d\theta=a, \tag{22a}$$

$$a_p+\int_0^{\pi}[b(\theta)e^{ip\theta}+b^*(\theta)e^{-ip\theta}]e^{-\lambda(\theta)}d\theta$$
$$=a_{p+1}+\int_0^{\pi}[b(\theta)e^{i\theta}e^{ip\theta}+b^*(\theta)e^{-i\theta}e^{-ip\theta}]d\theta. \tag{22b}$$

We can define $b(\theta)$ over the entire interval from 0 to 2π through the reality condition

$$b(2\pi-\theta)=b^*(\theta) \quad (0 \leqslant \theta \leqslant \pi). \tag{23}$$

Then, using Eqs. (18a) and (18b) in Eq. (22b) one finds after some algebra,

$$b(\theta)=-\frac{1}{2\pi}\frac{1}{2D+1}\frac{1}{1-\nu\cos\theta}\frac{1}{1-e^{-i\theta}e^{-\mu(1-\nu\cos\theta)}}, \tag{24}$$

where we have introduced for notational convenience

$$\mu \equiv \frac{2D+1}{\chi}, \tag{25a}$$

$$\nu \equiv \frac{2D}{2D+1}=\frac{2}{\gamma+\gamma^{-1}} \quad (<1). \tag{25b}$$

In arriving at Eq. (24) we have used the matching conditions (22b) and have thereby obtained all the ingredients of the exact integral expression (19) excepting the "speed" χ (or equivalently, μ) for which we use Eq. (22a) to obtain

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{(1-\nu\cos\theta)(1-e^{-i\theta}e^{-\mu(1-\nu\cos\theta)})}$$
$$=-(2D+1)(a-a_0), \tag{26}$$

which determines μ (hence χ) implicitly.

As an application of the basic formulas (24) and (26), we now consider two opposite limits relating to a propagating kink: propagation with infinitely large speed and that with vanishingly small speed. The latter is referred to as propagation failure or *pinning*. With D fixed, we consider $\chi \to 0$, i.e., $\mu \to \infty$, in which case the integral in Eq. (26) can be evaluated [using Eq. (18a) with $p=0$] and one finds that for given values of the coupling constant D and recovery parameter w pinning occurs as the threshold parameter a is made to approach the limiting value (the "pinning limit")

$$\tilde{a}=-w+\frac{1}{2}\left[1-\sqrt{\left(\frac{1}{1+4D}\right)}\right], \tag{27}$$

confirming the result obtained by Fath [11] who calculated the pinning threshold in the special case $w=0$. The opposite limit $\chi \to \infty$ $(\mu=0)$ cannot, however, be taken in Eq. (26) in a straightforward manner since the resulting integrand is singular at $\theta=0$. However, separating the contribution of the pole from the principal value one can evaluate the integral easily and finds that the pulse speed goes to ∞ in the limit

$$\frac{2D+1}{1+\gamma}=-(2D+1)(a-a_0) \tag{28a}$$

i.e.,

$$a \to -w. \tag{28b}$$

Having obtained the two limits, we seek for asymptotic expressions for small and large μ respectively. The left hand side of Eq. (26) [we denote this by $I(\mu,D)$] is first converted to a contour integral over the unit circle

$$I(\mu,D)=\frac{1}{2i\pi}\sum_n \oint\frac{dz}{z}\frac{z^{-n}}{1-\dfrac{z+z^{-1}}{\gamma+\gamma^{-1}}}e^{-n\mu}e^{(n\mu\nu/2)[z+(1/z)]}, \tag{29}$$

in which the last factor is the generating function for the Bessel functions with imaginary argument [12]

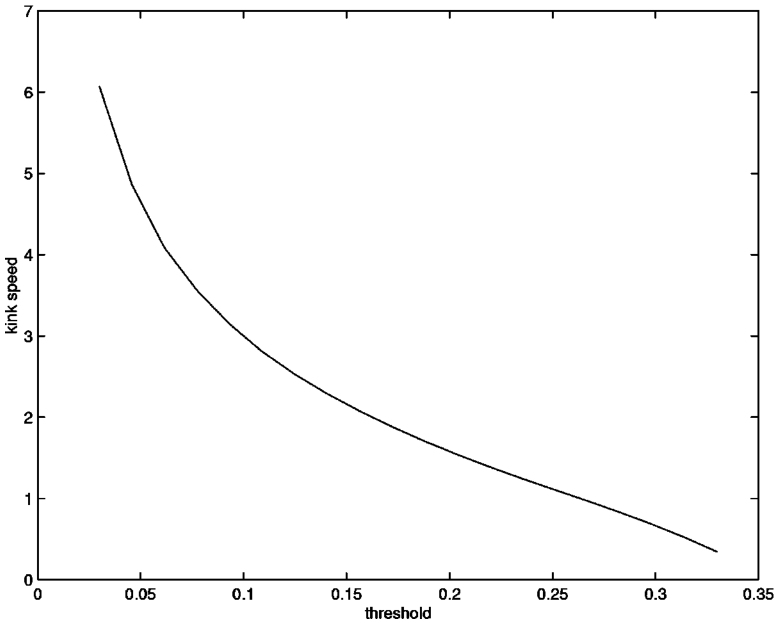

FIG. 1. Variation of kink speed χ with threshold parameter a, as computed numerically from Eq. (26); $D=2.0$, $w=0$; for these parameter values, the pinning limit is $\tilde{a}=0.3333$ [Eq. (27)] while the kink speed goes to infinity as $a\rightarrow 0$.

$$I_r(n\mu\nu)=e^{-ri\pi/2}J_r(in\mu\nu). \tag{30}$$

The residues at the poles $z=\gamma$ and $z=0$ can now be evaluated to yield

$$I(\mu,D)=\frac{1+\gamma^2}{1-\gamma^2}\sum_{n=0}^{\infty} e^{-n\mu}\left(I_n(n\mu\nu)+\sum_{p=1}^{\infty}\gamma^p[I_{n+p}(n\mu\nu)+I_{n-p}(n\mu\nu)]\right). \tag{31}$$

Substituting Eq. (31) in Eq. (26) we obtain another exact form of the equation determining the speed of the traveling kink.

As an application of Eq. (31) we give below the asymptotic expression for $I(\mu,D)$ with $\mu\approx 0$ $(a+w\approx 0)$ by using the series expansion for $I_r(x)$, which thereby leads to the corresponding kink speed χ

$$I(\mu,D)=\frac{1+2D}{1+\gamma}-\frac{\mu^2 D}{2(2D+1)}+O(\mu^3), \tag{32a}$$

$$\chi\approx\sqrt{\left(\frac{D}{2(a+\omega)}\right)}. \tag{32b}$$

In a similar manner, we look for the asymptotic expression for the speed close to the pinning limit (27). Since μ is large here, we invoke the asymptotic expression for Bessel functions with large argument. Noting that $\nu<1$, it is easily seen that the term with $n=0$ in Eq. (31) dominates over all other terms, the next approximant being the one with $n=1$. With these two terms retained in the sum, one obtains, for large μ,

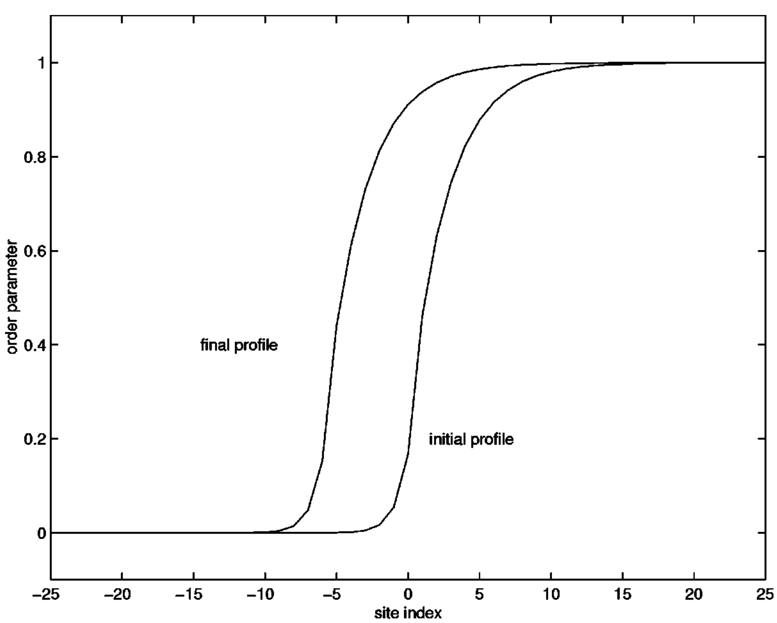

FIG. 2. Kink propagation. The initial profile was computed from Eqs. (19) and (24) for $D=2$, $w=0$, $a=0.1667$. The system was then allowed to evolve by Eqs. (1) and (2) up to $t=3.0$; the final profile is found to be simply a translated version of the initial profile consistent with Eq. (19) and the amount of translation is exactly χt where $\chi=1.9472$ as obtained from Eq. (24).

$$a=-w+\frac{\gamma}{1+\gamma}-\frac{e^{-\mu(1-\nu)}}{\sqrt{2\pi\mu\nu}}, \tag{33}$$

and, using Eq. (27) for the pinning limit ($\tilde{a}$) of the threshold parameter a, one gets, for $a\approx\tilde{a}$,

$$\sqrt{\chi}e^{-1/\chi}=\sqrt{4\pi D}(\tilde{a}-a), \tag{34}$$

in conformity with the result obtained by Fath [11].

Figure 1 shows the variation of kink speed with threshold a (for fixed D, w) as computed from Eq. (26), while Fig. 2 depicts the propagation of an initial profile chosen in accordance with Eqs. (19) and (24). The profile was made to evolve through Eqs. (1) and (2) for a given time and one finds that the final profile is indeed a translated version of the initial one, the amount of translation being entirely consistent with the speed computed from Eq. (26).

In summary, we have obtained an exact integral expression for the traveling kink solution in a PWL version of the discrete Nagumo equation and have derived an exact implicit relation determining the kink speed. Numerical integration of the evolution equation as also a recent work by Fath [11] corroborate these results. The inclusion of a recovery parameter in the model allows one to obtain pulse solutions as well. This will be reported in a future communication.

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